Problem 1 (20 Points)

a. (10) Using truth tables, show that

$(p \to r) \wedge (q \to r)$ is equivalent to $(p \vee q) \to r$

~~Use truth tables to show that the given compound proposition is a tautology.~~

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$p \vee q$	$(p \vee q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

b. (10) Using (a), and without using truth tables, show that the following is a tautology, using equivalences. Justify each step in your proof.

$$(p \vee q) \wedge (p \to r) \wedge (q \to r) \;\to\; r$$

$$\equiv (p \vee q) \wedge ((p \vee q) \to r) \;\to\; r \quad \text{(part (a))}$$

$$\equiv T \quad \text{(Modus Ponens: } P \wedge (P \to r) \;\to\; r \text{ is a tautology with } P \text{ being } p \vee q \text{)}$$

Can also be done by using $p \to q \equiv \neg p \vee q$ several times, and then simplifying the expression to show that it is equivalent to T.

Problem 2 (15 Points)

Consider the following predicates:

$S(x)$: "x is a student in this class",
$J(x)$: "x is a student who knows how to write Java programs",
$C(x)$: "x is a student who knows how to write C++ programs".
$P(x)$: "x is a student who can get a well-paying job".

Where x is a variable whose domain is the set of students in the university.
Express each of the following in predicate logic:

a. (5) Salwa is a student who is not in this class, but knows how to program in C++.

$\neg S(Salwa) \wedge C(Salwa)$

b. (5) There are students in this class that do not know how to program in both Java and C++

$\exists x\ (S(x) \wedge (\neg J(x) \vee \neg C(x)\ )$

OR

$\exists x\ (\ S(x) \wedge\ \neg(J\ (x) \wedge C(x)\ )\ )$

Another answer which reflects another meaning of the sentence (English is ambiguous) is

$\exists x\ (S(x) \wedge \neg J(x) \wedge \neg C(x)\ \ )$

Both answers have been considered correct.

c. (5) There are at least two students in this class who know how to program in Java

$\exists x\ \exists y\ (\ S(x) \wedge S\ (y) \wedge J(x) \wedge J(y) \wedge J(x) \wedge\ \neg(x{=}y\ )\ \ )$

Problem 3 (15 Points)

Consider the following argument involving the predicates above of the previous problem:

$$\forall x\, (\; S(x) \rightarrow J(x) \vee C(x) \;)$$
$$S(Amin) \;\wedge\; \neg C(Amin)$$
$$\forall x\, (\; J(x) \rightarrow P(x) \;)$$

$$\therefore\;\; \exists x\, (\, S(x) \wedge P(x) \,)$$

a. (10) Establish the validity of the argument justifying each step.

1.	$\forall x\, (\; S(x) \rightarrow J(x) \vee C(x) \;)$	Premise
2.	$S(Amin) \rightarrow J(Amin) \vee C(Amin)$	(1) and Universal Instantiation (UI)
3.	$S(Amin) \;\wedge\; \neg C(Amin)$	Premise
4.	$S(Amin)$	(4), Simplification
5.	$J(Amin) \vee C(Amin)$	(2), (4) and Modus Ponens
6.	$\neg C(Amin) \wedge S(Amin)$	(3), $\wedge$ is Commutative
7.	$\neg C(Amin)$	(6), Simplification
8.	$C(Amin) \vee J(Amin)$	(5), $\vee$ is Commutative
9.	$J(Amin)$	(8), (7), Disjunctive Syllogism
10.	$S(Amin) \;\wedge\; J(Amin)$	(4), (9) and Conjunction
11.	$\therefore\;\; \exists x\, (\, S(x) \wedge P(x) \,)$	(10), Existential Generalization

b. (5) Express in English the argument above.

Any student in the class knows how to program in Java or in C++. Amin is a student in the class who does not know how to program in C++. Any student who knows how to program in Java can get a well-paid job. Therefore, some student in the class can get a well-paid job.

Problem 4 (15 Points)

Recall that a real number x is **rational** if it can be written as the quotient of two integers; i.e. $x = m/n$, where m and n are integers, n non-zero; Otherwise, x is said to be **irrational**.

Prove or disprove each of the following. In each case specify which method of proof did you use.

a. (5) The product of two rational numbers is rational.

Direct Proof:

Let x and y be any two rational numbers.
Then $x = \frac{m}{n}$ and $y = \frac{p}{q}$ for some integers $m, n, n \neq 0$, and $p, q, q \neq 0$.
So $xy = = \frac{mp}{nq}$. mp and nq are integers, with $nq \neq 0$ (since $n \neq 0$, and $q \neq 0$)
Therefore xy is rational.

b. (5) The sum of two irrational numbers is irrational.

Counter Example:

Let $x=\sqrt{2}$, and $y = -\sqrt{2}$. Then x and y are irrational. Yet their sum is 0, $x + y = \sqrt{2} +(-\sqrt{2}) = 0$.
And 0 is rational.

c. (5) The sum of a rational number and an irrational number is irrational.

Proof by Contradiction:

Let x be a rational number, $x = \frac{m}{n}$ with integers $m, n, n \neq 0$, and let y irrational. Suppose that $x+y=z$.
For the sake of contradiction, assume z is rational. Then $z = \frac{p}{q}$ with integers p $,q, q \neq 0$.
Thus $y = z – x = \frac{m}{n} - \frac{p}{q} = \frac{mq-np}{nq}$
But mq-np is an integer, and nq is an integer, $nq \neq 0$ (since $n \neq 0$, and $q \neq 0$)

So y is rational.

Contradiction (y is irrational and y is rational).

So our assumption that z is rational is false. So z is irrational.

Problem 5. (10 Points)

Suppose $B = \{\ 2, \{2\}\ \}$. Mark each of the following statements as TRUE or FALSE (Circle the right answer)

a.	$\{2\} \in B$.	**TRUE**	FALSE
b.	$\{2\} \subseteq B$.	**TRUE**	FALSE
c.	$2 \subseteq B$.	TRUE	**FALSE**
d.	$\lvert B \rvert = 1$	TRUE	**FALSE**
e.	$\phi = \{\phi\}$	TRUE	**FALSE**

Problem 6. (10 Points)

Suppose that the set membership table for a set expression *X* involving two sets *A* and *B* is the following:

A	*B*	X
1	1	0
1	0	1
0	1	0
0	0	1

Give a set expression for *X*, in terms of *A* and *B* (or their complements!)

Similar to Distributive Normal Form adapted to Set Membership. Then have a union of intersections: See where there is a 1 in the column of *X*; then form the intersection of *A* or its complement with *B* or its complement, depending on whether *A*=1 (then include *A*) or 0 (then include *A'*s complement), and the same for *B*.

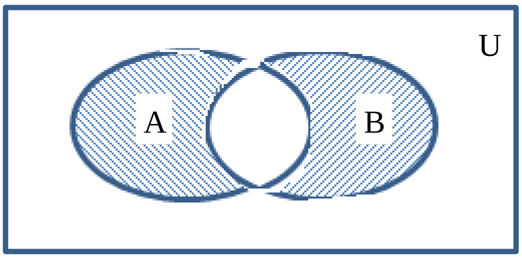
So here: $X = (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B})$

Problem 7 (25 Points)

The symmetric difference of two sets A and B denoted by $A \oplus B$ is the set of elements that are in A or in B but not in both!! So $\{1, 3, 5\} \oplus \{1, 2, 3\} = \{2, 5\}$.

$$x \in A \oplus B \text{ is equivalent to } x \in (A \cup B) \wedge x \notin (A \cap B)$$

a. (5) Draw a Venn diagram to represent $A \oplus B$.



b. (10) Use set builder notation and propositional logic to show that $A \oplus B = (A\text{-}B) \cup (B\text{-}A)$

$A \oplus B = \{ x \mid x \in (A \cup B) \wedge x \notin (A \cap B) \}$ by definition

$= \{ x \mid (x \in A \vee x \in B) \wedge \neg (x \in A \wedge x \in B) \}$

$= \{ x \mid (x \in A \vee x \in B) \wedge (\neg x \in A \vee \neg x \in B) \}$ De Morgans

$= \{ x \mid ((x \in A \vee x \in B) \wedge \neg x \in A) \vee ((x \in A \vee x \in B) \wedge \neg x \in B) \}$ Distributive property of $\wedge$ wrt $\vee$

$= \{ x \mid ((x \in A \wedge \neg x \in A) \vee (x \in B \wedge \neg x \in A)) \vee ((x \in A \wedge \neg x \in B) \vee (x \in B \wedge \neg x \in B)) \}$

Distributive property of $\wedge$ wrt $\vee$

$= \{ x \mid (F \vee (x \in B \wedge \neg x \in A)) \vee ((x \in A \wedge \neg x \in B) \vee F) \}$ Negation laws $p \wedge \neg p \equiv F$

$= \{ x \mid ((x \in B \wedge \neg x \in A)) \vee ((x \in A \wedge \neg x \in B)) \}$ Identity law $p \vee F \equiv p$

$= \{ x \mid ((x \in B\text{-}A)) \vee ((x \in A\text{-}B)) \}$ Definition of $B\text{-}A$ and $A\text{-}B$

$= (B\text{-}A) \cup (A\text{-}B)$ Definition of $\cup$

$= (A\text{-}B) \cup (B\text{-}A)$ $\cup$ is commutative

c. (10) Show that $A = B$ if and only if $A \oplus B = \phi$. (Hint: Use (b), and if $A\text{-}B = \phi$ then $A \subseteq B$)

Suppose $A \oplus B = \phi$, then $(A\text{-}B) \cup (B\text{-}A) = \phi$. So $A\text{-}B = \phi$ and $A\text{-}B = \phi$. So $A \subseteq B$ and $B \subseteq A$. So $A = B$.

Now, suppose $A = B$. Then $A\text{-}B = \phi$ and $A\text{-}B = \phi$. So $(A\text{-}B) \cup (B\text{-}A) = \phi$. Therefore $A \oplus B = \phi$.

Problem 1 (15 Points)

Let A and B denote sets of numbers, and let $f : A \rightarrow B$ be a function such that $f(x) = x^2$.

a. (10) In the table below, each row corresponds to a certain choice of the domain A and the codomain B and hence to a certain choice of the function f..
The sets chosen are from the following:

$\mathbb{N}$, denotes the set of natural numbers, i.e. {0, 1, 2, ….}
$\mathbb{Z}$, denotes the set of integers (whole numbers)
$\mathbb{R}$, denotes the set of real numbers
$\mathbb{R}^+$, denotes the set of non-negative real numbers (including 0)

Fill in the table below: For each row, put Yes or No to indicate whether the function is One-to-One and/or Onto. Also specify the Range of f.

A	B	One-to-one	Onto	Range of f
$\mathbb{N}$	$\mathbb{N}$	Yes	No	$\{n \in \mathbb{N} : n$ is a perfect square $\}$ $\{0, 1, 4, 9, 16, 25, \ldots\}$
$\mathbb{Z}$	$\mathbb{N}$	No	No	$\{n \in \mathbb{N} : n$ is a perfect square $\}$ $\{0, 1, 4, 9, 16, 25, \ldots\}$
$\mathbb{R}$	$\mathbb{R}^+$	No	Yes	$\mathbb{R}^+$
$\mathbb{R}^+$	$\mathbb{R}^+$	Yes	Yes	$\mathbb{R}^+$

b. (5) Among the above cases, specify the cases in which the function is invertible, and specify the inverse in that case.

It is the case corresponding to the last row: $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,with $f(x) = x^2$ .In this case, f is a bijection. So f is invertible, and : $f^{-1} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,with $f^{-1}(x) = \sqrt{x}$

Problem 2 (10 Points)

Let $f: A \rightarrow B$, and $g: B \rightarrow C$. The *composition of f with g*, denoted by $g \circ f$ is a function from A to C, where $g \circ f(x) = g(f(x))$.

a. (6) Show that if $g \circ f$ is one-to-one, then f is one-to-one.

Suppose $f(a_1) = f(a_2)$.

Then $g(f(a_1)) = g(f(a_2))$. So $g \circ f(a_1) = g \circ f(a_2)$. But $g \circ f$ is a one-to-one function.

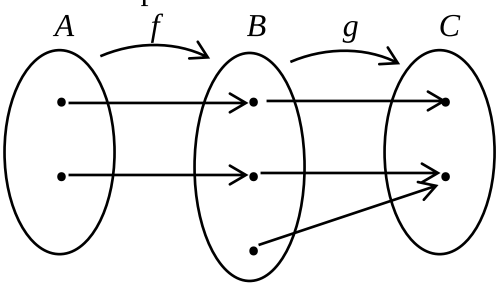
So $a_1 = a_2$

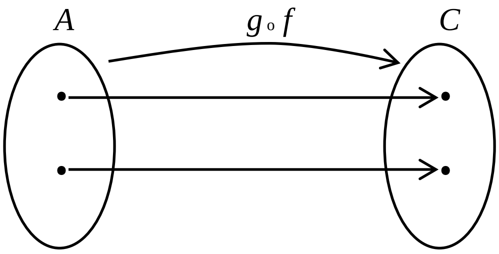
Thus f is one-to-one.

b. (4) Show that if $g \circ f$ is one-to-one, then g need not be one-to-one (Give an example)

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,with $f(x) = x$, and $g : \mathbb{R} \rightarrow \mathbb{R}^+$,with $g(x) = x^2$.
(f is one-to-one and onto, g is ***not*** one-to-one, but it is onto).
Then $g \circ f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,with $g \circ f(x) = x^2$ which is one-to-one (and onto)..

Another example:





Problem 3 (15 Points) Floor and Ceiling Functions

a. (5) Fill in the following table

x	$\lfloor x \rfloor$	$\lceil x \rceil$
0.2	0	1
- 0.2	-1	0
-4	-4	-4

b. (5) True or False: If true, prove it, otherwise give a counter example:
$\lfloor x \rfloor + \lceil x \rceil = 2 \lfloor x \rfloor$, for any real number x

False. Counter example: Let $x = 0.5$.
Then $\lfloor x \rfloor = 0$, $\lceil x \rceil = 1$. So $\lfloor x \rfloor + \lceil x \rceil = 1$, while $2 \lfloor x \rfloor = 0$
So $\lfloor x \rfloor + \lceil x \rceil \neq 2 \lfloor x \rfloor$

c. (5) True or False: If true, prove it, otherwise give a counter example:
$\lfloor x + 0.5 \rfloor = \lceil x - 0.5 \rceil$, for any real numbers $x \geq 0$

False. Counter example: Let $x = 1.5$, so $x + 0.5 = 2$, and $x - 0.5 = 1$.
Then $\lfloor x + 0.5 \rfloor = 2$, while $\lceil x - 0.5 \rceil = 1$.
So $\lfloor x + 0.5 \rfloor \neq \lceil x - 0.5 \rceil$

Problem 4 (15 Points)

a. (10) Use mathematical induction to prove that $\sum_{i=1}^{n}(4i-1) = n(2n+1)$ for $n \geq 1$.

Basis Step: $n = 1$.
$\sum_{i=1}^{1}(4i-1) = 4 - 1 = 3$. Also $n(2n+1) = 1(2+1) = 3$.

Inductive Step.
Assume $\sum_{i=1}^{k}(4i-1) = k(2k+1)$ for $k \geq 1$. We show that $\sum_{i=1}^{k+1}(4i-1) = (k+1)(2(k+1)+1)$

$\sum_{i=1}^{k+1}(4i-1) = \sum_{i=1}^{k}(4i-1) + 4(k+1) - 1 = k(2k+1) + 4(k+1) - 1$ (by induction hypothesis)
$= 2k^2 + k + 4k + 4 - 1$
$= 2k^2 + 5k + 3$

On the other hand,
$(k+1)(2(k+1)+1) = (k+1)(2k+3) = 2k^2 + 2k + 3k + 3 = 2k^2 + 5k + 3$.
So they are equal!!

Thus by the Principle of Mathematical Induction, the result follows.

b. (5) Prove the same result above by splitting the sum, and using $1+2+\ldots+n = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n}(4i-1) = \sum_{i=1}^{n} 4i - \sum_{i=1}^{n}(1) = 4\sum_{i=1}^{n} i - \sum_{i=1}^{n}(1)$
$= 4(1+2+\ldots+n) - (1 + 1 + \ldots + 1)$
$= 4\frac{n(n+1)}{2} - n$
$= 2n(n+1) - n$
$= n(2n + 2 - 1) = n(2n+1)$.

Problem 5 (20 Points)

In this problem you are requested to give algorithms that will check if a list of numbers is not in order (not sorted) and "to what degree"

Use pseudo code to write your algorithms. You may use corresponding simple syntax provided by a programming language that you may be familiar with like Java, C++, C, but you not use sophisticated constructs and/or structures that may be provided. You are not allowed to use any built-in functions or data structures including arrays. For example: no **substring**, no **String a[]**, no **LinkesList**, etc. You can use **for** loops, **while** loops, variable declarations, assignment statements, simple comparison statements (**if**, **if-else**, but no **switch** statement). Certainly no pointers!You can use curly braces **{ }** to indicate scope, or explicit lexemes like **begin** and **end**, but still indent your code to make it easier to read.

a. (7) Write an algorithm *Out-of-Order* that takes as input a list $a_1, a_2, \ldots, a_n$ and returns the *location* of the first entry of the list which is out of order (if any). Assume that the list is supposed to be in increasing order. The algorithm should return 0 if the list is in sorted. To illustrate, if the list is (8, 5, 12, 10), your algorithm should return 2. If the list is (2, 5, 1, 10) your algorithm should return 3. If the list is (2, 5, 8, 10) your algorithm should return 0.

procedure *Out-of-Order* ($a_1, a_2, \ldots, a_n$: integers)
    $k := 1$
    while ($k < n$ **and** $a_k < a_{k+1}$)
        $k := k + 1$
    if $k<n$ **then** *location* := k +1
else *location* := 0
return *location*{*location* is the subscript of the first term that is not in order, or is 0 if all is in order}

b. (6) Write an algorithm *Count* that returns the number of violations of the condition $a_i < a_j$ in a given list $a_1, a_2, \ldots, a_n$ that is supposed to be in increasing order. To illustrate, if the list is (8, 5, 12, 10), then *count* should be returned as 2. If the list is (2, 5, 1, 10) your algorithm should return 1. If the list is (2, 5, 8, 10) your algorithm should return 0.

```
procedure Count (a1, a2, …., an : integers)
    count := 0
    for i := 1 to n-1
        if ai ≥ ai+1 then count := count + 1
    return count {count is the number of violations}
```

c. (7) Write an algorithm *Both* that combines the objectives in (a) and (b) above to return a pair of values *location* and *count*. The algorithm should go through the list only one time.

```
procedure Both (a1, a2, …., an : integers)
    k := 1
    count := 0
    while (k < n and ak < ak+1)
        k := k + 1
    if k<n then
        location := k +1
        count := 1
        for i := k+1 to n-1
                if ai ≥ ai+1 then count := count + 1
    else location := 0
    return location , count
```

Problem 6 (15 Points)

Consider the proposition *P(n)*: An amount of postage of *n* cents can be formed using 3-cent and 4-cent stamps.

a. (5) Find the smallest *m* such that *P(n) is true* for all $n \geq m$

$m = 6$.

Conjecture: *P(n) is true* for all $n \geq 6$

i.e. want to show that for any $n \geq 6$, we can find natural numbers *a,b* such that $n = 3a + 4b$

Note *P*(5) is not true…

That is why 6 is the minimum

b. (5) Use mathematical induction to prove your conjecture in (a) that *P(n) is true* for $n \geq m$.

Basis Step: $n=6$.

$6 = 3*2 + 4*0$

Thus *P*(6) is true with witnesses $a=2$, and $b=0$.

Inductive step: Assume *P*(*k*) is true, for some $k \geq 6$. We show that *P*(*k*+1) is true.

Since *P*(*k*) is true, then $\exists\, a,b \in \mathbb{N}$, such that $k = 3a + 4b$.

Case 1. $a=0$. Then $k = 4$b. Since $k \geq 6$, then *b* is at least 2; i.e. $b \geq 2$.

$k+1 = 4b + 1 = 4(b-2) + 8 + 1 = 4(b-2) + 9 = 3*3 + 4(b-2)$.

Thus *P*(*k*+1) is true with witnesses 3 and *b*-2.

Case 2. $a \neq 0$.

Then $k = 3a + 4b$. So

$k+1 = 3a + 4b + 1 = 3(a-1) + 3 + 4b +1 = 3(a-1) + 4b +4 = 3(a-1) + 4(b +1)$.

Thus *P*(*k*+1) is true with witnesses *a*-1 and *b*+1.

So by the Principle of Mathematical Induction, *P*(*n*) is true for all *n* $n \geq 6$.

c. (5) Use strong mathematical induction to prove the same result.

Basis Step:

P(6) is true: $6 = 3*2 + 4*0$ (witnesses 2 and 0)

P(7) is true: $7 = 3*1 + 4*1$ (witnesses 1 and 1)

P(8) is true: $8 = 3*0 + 4*2$ (witnesses 2 and 0)

Inductive Step: Assume $P(k-2) \wedge P(k-1) \wedge P(k)$ is true. We show *P*(*k*+1) is true.

In fact here we only need *P*(*k*-2) to be true. Suppose $k-2 = 3a + 4b$. Then

$k+1 = k - 2 + 3 = 3a +4b + 3 = 3(a+1) + 4b$.

Thus *P*(*k*+1) is true with witnesses *a*+1 and *b*.

So by the Principle of Strong Mathematical Induction, *P*(*n*) is true for all *n* $n \geq 6$.

Problem 7 (15 Points)
In this problem, all functions are on positive natural numbers.
True or false. If true, prove it and give the witnesses; otherwise give a counter example..

a. (5) Show that $log\ n$ is $O(n)$, using the fact that $n < 2^{n}$, for all integers $n > 0$. Make sure to give the witnesses.

Since $n < 2^{n}$, for all integers $n > 0$, and since *log* is an increasing function, we can take *log* on both sides and so we get $log\ n < n$, so $log\ n < Cn$, for all $n > k$, with k=0, and C=1.
Note: Here we have assumed that *log* is base 2. Otherwise, C will change.

b. (5) Suppose that $f(n)$ is $O(n^3)$. Let $g(n) = f(n) + n^2\ log\ n$. Using (a), show that $g(n)$is $O(n^3)$. Make sure to give the witnesses.

Given $f(n)$ is $O(n^3)$, so $|\ f(n)| \leq C_1 n^3$, for $n > k_1$, for some C_1 and k_1.
From part (a), $log\ n < n$ for n>0. So $n^2\ log\ n < n^3$, n>0.

Thus $|\ g(n)\ | = \ |\ f(n) + n^2\ log\ n\ | \ \leq |\ f(n)\ | \ + n^2\ log\ n$
$\leq C_1 n^3 + n^3$, for $n > max(k_1,0) = k_1$
$= (1+ C_1)\ n^3$, for $n > k_1$
Thus $g(n)$is $O(n^3)$ with witnesses $k = k_1$, and $C = 1 + C_1$

c. (5) True or False. Suppose that $f(n)$ is $O(n^3)$. Then $f(n)$ **cannot** be $O(n^2)$

False. If $f(n) = n^2$, then $f(n)$ is $O(n^2)$ and $f(n)$ is $O(n^3)$

Problem 8 (15 Points) Greedy Algorithm

Consider the greedy change-making algorithm for n cents. The algorithm works with any coin denominations $c_1, c_2, \ldots, c_r$

procedure *change*($c_1, c_2, \ldots, c_r$: values of coins, where $c_1 > c_2 > \ldots > c_r$; n: a positive integer)
for $i := 1$ to r
$d_i := 0$ [d_i counts the coins of denomination c_i]
while $n \geq c_i$
$d_i := d_i + 1$ [add a coin of denomination c_i]
$n = n - c_i$
[d_i counts the coins c_i]

For the example of U.S. currency, where we have quarters, dimes, nickels and pennies, with $c_1 = 25$, $c_2 = 10$, $c_3 = 5$, and $c_4 = 1$. Apply the algorithm with $n = 86$.

a. (5) What will be the values returned in the d_i's upon running *change*(25, 10, 5, 1, 86)

$d_1 = 3$, $d_2 = 1$, $d_3 = 0$, and $d_4 = 1$

b. (5) List the successive values of n during the run in (a).

n = 86 then 61= (86-25), then 36= (61-25) then 11= (36-25)
then 1= (11-10)
then 0= (1-1)

c. (5) Zakkour runs the algorithm on a certain n, other than 86, and he claims the following results:
$d_1 = 1$, $d_2 = 3$, $d_3 = 1$, and $d_4 = 3$,

What is n?
n would be 1*25 + 3*10 + 1*5 + 3*1 = 63

Are the results of Zakkour optimal? Why?
No it is not optimal!! The number of coins as claimed is 1+3+1+3 = 8

In fact $d_2 = 3$ is problematic (not optimal), as 3*10 = 30, which could be replaced by 1*25 + 1*5 with 2 coins (1 quarter and 1 nickel) instead of 3 (dimes). Subsequently, the 2 nickels can be replaced by one dime.

The optimal solution is: 2*25 + 1*10 + 0*5 + 3*1 = 63, where the number of coins is 2+1+0+3=6 (instead of 8)